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ASYMPTOTIC METHODS IN THE MECHANICS OF CONTINUOUS MEDIA: PROBLEMS WITH MIXED BOUNDARY CONDITIONS†

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Regular and singular asymptotic methods are applied to one- and two-dimensional integral equations of the first kind that arise in the treatment of various two-dimensional axisymmetrical and three-dimensional problems with mixed boundary conditions in the mechanics of continuous media.

ASYMPTOTIC methods have several advantages: universality, the analytical form of the solutions obtained and the simplicity of further qualitative and quantitative analysis. Since problems with mixed boundary conditions can usually be reduced to the solution of integral equations, the latter are the real object of attention in this paper, where it is proposed to use asymptotic methods to investigate them. Later a few typical integral equations (IEs) will be considered [1–11].

1. TWO-DIMENSIONAL PROBLEMS AND A FEW THREE-DIMENSIONAL PROBLEMS

Consider the IE

$$\int_{-1}^{1}\varphi(\xi)k\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi f(x)$$
$$|x|\leqslant 1,\quad \lambda\in(0,\infty),\quad f(x)\in H_1^{\alpha}(-1,1),\quad \alpha>\frac{1}{2} \tag{1.1}$$

$$k(t)=\int_0^{\infty}\frac{L(u)}{u}\cos ut\,du \tag{1.2}$$

($H_m^{\alpha}(-\beta,\beta)$ is the space of functions whose mth derivatives satisfy a Hölder condition with exponent α for $|x|\leqslant\beta$). The function $L(u)$ is continuous and positive for $u\in(0,\infty)$, and it satisfies the following asymptotic equalities

$$L(u)=Au+O(u^3)\quad (u\to 0,\quad A=\text{const}>0) \tag{1.3}$$
$$L(u)=1+\sum_{i=1}^{N-1}\frac{B_i}{u^i}+O\left(\frac{1}{u^N}\right)\quad (u\to\infty,\quad B_i=\text{const})$$

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We shall also assume that $L(u)u^{-1}$ and $u[L(u)]^{-1}$, being functions of the complex variable $w=u+iv$, are regular in strips $|v|\leqslant\gamma_1$ and $|v|\leqslant\gamma_2$, respectively. Hence it follows, in particular, that the kernel $k(t)$ decreases at infinity at least as rapidly as $\exp(-\gamma_1|t|)$.

We consider the Hilbert space $H(-1, 1)$ with norm

$$\|\varphi\|_H^2=\int_{-1}^{1}\int_{-1}^{1}\varphi(x)\varphi(\xi)k\left(\frac{\xi-x}{\lambda}\right)dxd\xi=\frac{1}{2}\int_{-\infty}^{\infty}\frac{L(\alpha\lambda)}{\alpha}|\Phi(\alpha)|^2 d\alpha \tag{1.4}$$

where $\Phi(\alpha)$ is the Fourier transform of the function $\varphi^*(x)$ ($\varphi^*(x)=\varphi(x)$ for $|x|\leqslant 1$ and $\varphi^*(x)=0$ for $|x|>1$). Using Riesz's theorem on the form of bounded linear functionals, one can show that the solution of IE (1.1), (1.2) in $H(-1, 1)$ exists and is unique for any $\lambda\in(0, \infty)$; it is in fact

$$\varphi(x)=\omega(x)(1-x^2)^{-1/2},\quad \omega(x)\in C(-1,1) \tag{1.5}$$

The equation is well posed, in the sense that

$$\|\omega\|_C\leqslant m\|f\|_{H_1^\alpha} \tag{1.6}$$

From (1.3) we obtain a representation for the kernel (1.2)

$$k(t)=\sum_{i=0}^{\infty}a_i t^{2i}+|t|\sum_{i=0}^{\infty}b_i t^{2i}+\ln|t|\sum_{i=0}^{\infty}c_i t^{2i} \tag{1.7}$$

where $c_0=-1$ and the series converges uniformly for $|t|<\rho$, $\rho\leqslant\infty$. It follows from the structure (1.7) of $k(t)$ that for sufficiently large λ the solution of IE (1.1) may be written as

$$\varphi(x)=\sum_{i=0}^{N-i}\sum_{j=0}^{i}\varphi_{ij}(x)\lambda^{-i}(\ln\lambda)^j+O[\lambda^{-N}(\ln\lambda)^N] \tag{1.8}$$

Substituting (1.7) and (1.8) into (1.1), we obtain a system of IEs for the successive determination of the functions of the type

$$-\int_{-1}^{1}\varphi(\xi)\ln\left|\frac{\xi-x}{\lambda}\right|d\xi=\pi g(x)\quad(|x|\leqslant 1) \tag{1.9}$$

As such equations can be solved in closed form for any $g(x)\in H_1^\alpha(-1, 1)$, the regular asymptotic expansion (1.8) can actually be constructed to within any desired accuracy. For practical purposes it is usually sufficient to retain terms of the order of λ^{-4}, in which case the solution (1.8) of Eq. (1.1) will be valid throughout the range $\lambda\geqslant\sup(2, 2/\rho)$.

Consider the system of two IEs

$$\begin{aligned}\int_{-1}^{\infty}\varphi_1(\xi)k\left(\frac{\xi-x}{\lambda}\right)d\xi&=\pi f_1(x)+\int_{-\infty}^{-1}\varphi_2(\xi)k\left(\frac{\xi-x}{\lambda}\right)d\xi\quad(-1\leqslant x<\infty)\\ \int_{-\infty}^{1}\varphi_2(\xi)k\left(\frac{\xi-x}{\lambda}\right)d\xi&=\pi f_2(x)+\int_{1}^{\infty}\varphi_1(\xi)k\left(\frac{\xi-x}{\lambda}\right)d\xi\quad(-\infty<x\leqslant 1)\end{aligned} \tag{1.10}$$

The functions $f_1(x)$ and $f_2(x)$ are such that

$$\begin{aligned}&f_1(x)=O(e^{-\alpha_1 x})\quad(x\to\infty,\ \alpha_1>0)\\ &f_2(x)=O(e^{\alpha_2 x})\quad(x\to-\infty,\ \alpha_2>0)\\ &f_1(x)+f_2(x)=f(x)\quad(|x|\leqslant 1)\end{aligned} \tag{1.11}$$

When the last of conditions (1.11) is satisfied, the solution of IE (1.1) can be found as the sum of solutions of IEs (1.10), i.e.

$$\varphi(x) = \varphi_1(x) + \varphi_2(x) \quad (|x| \leqslant 1) \tag{1.12}$$

It can be shown that the first two conditions (1.11) imply

$$\begin{aligned} &\varphi_1(x) = O(e^{-\beta_1 x}) \quad (x \to \infty,\ \beta_1 > 0) \\ &\varphi_2(x) = O(e^{\beta_2 x}) \quad (x \to -\infty,\ \beta_2 > 0) \end{aligned} \tag{1.13}$$

If the function $f(x)$ in *IE* (1.1) is even or odd, then

$$f_1(x) = \pm f_2(-x), \quad \varphi_1(x) = \pm\varphi_2(-x) \tag{1.14}$$

In both these cases the system of IEs (1.10) reduces, via obvious substitutions, to a single IE

$$\begin{aligned} &\int_0^\infty \psi(\tau)k(\tau - t)d\tau = \pi h(t) \pm \int_{2/\lambda}^\infty \psi(\tau)k\left(\frac{2}{\lambda} - t - \tau\right)d\tau \quad (0 \leqslant t < \infty) \\ &\psi(t) = \varphi_1(\lambda t - 1), \quad h(t) = \lambda^{-1} f_1(\lambda t - 1) \end{aligned} \tag{1.15}$$

We shall always take the "plus" sign for the even case and the "minus" for the odd case.

It is obvious from expansion (1.7) that the kernel $k(t)$ has a logarithmic singularity at zero. In addition, it disappears exponentially at infinity. Taking these factors into account, together with the first relation of (1.13), one can show that the following asymptotic estimate holds uniformly in t

$$\int_{2/\lambda}^\infty \psi(\tau)k\left(\frac{2}{\lambda} - t - \tau\right)d\tau = O(e^{-2\beta_1/\lambda}) \tag{1.16}$$

By (1.16), the IE (1.15) can be solved for small λ by successive approximations, dropping the integral on the right in the zeroth approximation. When that is done, each iteration requires the solution of an IE of the form

$$\int_0^\infty \psi(\tau)k(\tau - t)d\tau = \pi l(t) \quad (0 \leqslant t < \infty) \tag{1.17}$$

Such IEs can be solved in closed form by the Wiener–Hopf method. One can therefore actually construct a singular asymptotic expansion of IE (1.1) for small λ as

$$\varphi(x) = \psi\left(\frac{1+x}{\lambda}\right) \pm \psi\left(\frac{1-x}{\lambda}\right) \tag{1.18}$$

to within any desired accuracy. For practical purposes, it is usually sufficient to consider the zeroth approximation, i.e. to take as $\psi(t)$ a solution of IE (1.17) for $l(t) \equiv h(t)$. This approximate solution holds throughout the range $\lambda \leqslant \sup(2, 2/\rho)$.

Thus, the regular asymptotic method for large λ and the singular asymptotic method for small λ cover the entire range of variation of λ guaranteeing a complete analytic solution of any problem that can be reduced to an IE of type (1.1), (1.2).

2. AXISYMMETRICAL PROBLEMS AND THE FIRST HARMONIC

Consider the IE

$$\int_0^1 \varphi(\rho)k_n\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)\rho d\rho=\lambda f(r)$$
$$0\leqslant r\leqslant 1;\quad \lambda\in(0,\infty);\quad n=0,1;\quad f(r)\in H_1^{1-0}(S) \tag{2.1}$$

$$k_n(\mu,\nu)=\int_0^\infty L(u)J_n(\mu u)J_n(\nu u)du \tag{2.2}$$

where S is a circle of unit radius, $J_n(x)$ the Bessel function and $L(u)$ has the properties described above.

Introducing the Hilbert space $H(S)$ with norm

$$\|\varphi\|_H^2=\frac{1}{\lambda}\int_0^1\int_0^1 \varphi(r)\varphi(\rho)k_n\left(\frac{\rho}{\lambda},\frac{r}{\lambda}\right)r\rho drd\rho=\int_0^\infty L(\alpha\lambda)\Phi^2(\alpha)d\alpha \tag{2.3}$$

where $\Phi(\alpha)$ is the Hankel transform of $\varphi^*(r)$ ($\varphi^*(r)=\varphi(r)$ in S and $\varphi^*(r)=0$ outside S), one can show, using Riesz's theorem, that IE (2.1) and (2.2) has a unique solution in $H(S)$ for any $\lambda\in(0,\infty)$, of the form

$$\varphi(r)=\omega(r)(1-r)^{-1/2},\quad \omega(r)\in C(S) \tag{2.4}$$

Moreover, the equation is well posed in the sense of (1.6).

The IE (2.1) and (2.2) for $n=0$ can be reduced to an equivalent IE

$$\int_{-1}^1 \psi(\xi)m\left(\frac{\xi-x}{\lambda}\right)d\xi=\pi\lambda g(x)\quad (|x|\leqslant 1) \tag{2.5}$$

$$m(t)=\int_0^\infty L(u)\cos ut du \tag{2.6}$$

where the even functions $\psi(x)$ and $g(x)$ are related to $\varphi(r)$ and $f(r)$ by the equations

$$\varphi(r)=-\frac{2}{\pi}\frac{d}{dr}r\int_r^1\frac{\psi(\xi)d\xi}{\xi\sqrt{\xi^2-r^2}},\quad g(x)=\frac{d}{dx}\int_0^x\frac{\rho f(\rho)d\rho}{\sqrt{x^2-\rho^2}} \tag{2.7}$$

For $n=1$ IE (2.1) and (2.2) may also be reduced to an equivalent IE (2.5) and (2.6) where now the odd functions $\psi(x)$ and $g(x)$ are related to $\varphi(r)$ and $f(r)$ by

$$\varphi(r)=-\frac{2}{\pi}\frac{d}{dr}\int_r^1\frac{\psi(\xi)d\xi}{\sqrt{\xi^2-r^2}},\quad g(x)=\frac{d}{dx}x\int_0^x\frac{f(\rho)d\rho}{\sqrt{x^2-\rho^2}} \tag{2.8}$$

Asymptotic solutions of IE (2.5) and (2.6) for large and small λ, together covering the entire range of variation of λ may be constructed along the same lines as was done above for IE (1.1) and (1.2). In particular, we note that, by (1.3), the kernel (2.6) may be represented as

$$m(t)=\pi\delta(t)+\sum_{i=0}^\infty \tilde{a}_i t^{2i}+|t|\sum_{i=0}^\infty \tilde{b}_i t^{2i}+\ln|t|\sum_{i=0}^\infty \tilde{c}_i t^{2i} \tag{2.9}$$

where the series are uniformly convergent for $|t|<\rho$, $\rho\leq\infty$, $\delta(t)$ being the delta-function. It is clear from (2.9) that for large λ a solution of IE (2.5) must again be sought in the form (1.8); but there is no need to solve an IE of the type (1.9)

We also note that, by [3, Theorem 41.2], a solution of Eq. (2.1) for any $n\geq 2$ may be constructed if one knows a solution of the equation for $n=0$.

3. THREE-DIMENSIONAL PROBLEMS

Consider the IE

$$\iint_\Omega \varphi(\xi,\eta)k\left(\frac{R}{\lambda}\right)d\xi d\eta = 2\pi\lambda f(x,y) \tag{3.1}$$

$$(x,y)\in\Omega,\quad R=\sqrt{(\xi-x)^2+(\eta-y)^2},\quad \frac{1}{2}\max_\Omega R=1,\quad f(x,y)\in H_1^{1-0}(\Omega)$$

$$k(t)=\int_0^\infty L(u)J_0(ut)du \tag{3.2}$$

The function $L(u)$ has the properties described above, the domain Ω is simply connected, and the contour L of Ω has continuous curvature.

Introducing the Hilbert space $H(\Omega)$ with norm

$$\|\varphi\|_H^2=\frac{1}{\lambda}\iint_\Omega\iint_\Omega \varphi(x,y)\varphi(\xi,\eta)k\left(\frac{R}{\lambda}\right)dxdyd\xi d\eta=$$

$$=\frac{1}{4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{L(\gamma\lambda)}{\gamma}|\Phi(\alpha,\beta)|^2 d\alpha d\beta \quad (\gamma=\sqrt{\alpha^2+\beta^2}) \tag{3.3}$$

where $\Phi(\alpha, \beta)$ is the two-dimensional Fourier transform of $\varphi^*(x, y)$ ($\varphi^*(x, y)=\varphi(x, y)$ in Ω, $\varphi^*(x, y)=0$ outside Ω) one can show, using Riesz's theorem, that IE (3.1), (3.2) has a unique solution in $H(\Omega)$ for any $\lambda\in(0, \infty)$, of the form

$$\varphi(x,y)=\omega(x,y)[N(x,y)]^{-1/2},\quad \omega(x,y)\in C(\Omega) \tag{3.4}$$

where $N(x, y)=0$ is the normalized equation [12] of the contour L, $N(x, y)>0$ in Ω. In addition, the equation is well posed in the sense of (1.6).

From (1.3) we obtain a representation for the kernel (3.2)

$$k(t)=t^{-1}+\sum_{i=0}^\infty a_i^* t^{2i}+t\sum_{i=0}^\infty b_i^* t^{2i}+\ln t\sum_{i=0}^\infty c_i^* t^{2i} \tag{3.5}$$

where the series converge uniformly for $t<\rho$, $\rho\leq\infty$. Substituting (3.5) into (3.1), we conclude that, for sufficiently large λ, the solution $\varphi(x, y)$ of IE (3.1) will have a form similar to (1.8), with $\varphi_{ij}(x)$ replaced by $\varphi_{ij}(x, y)$, if one can find a solution (even just an approximate solution) in Ω of the following IE, which is simpler than (3.1)

$$\iint_\Omega \frac{\varphi(\xi,\eta)}{R}d\xi d\eta=2\pi g(x,y),\quad (x,y)\in\Omega,\quad g(x,y)\in H_1^{1-0}(\Omega) \tag{3.6}$$

For practical purposes it is usually sufficient to truncate the series for $\varphi(x, y)$ by omitting terms of order higher than λ^{-4}; this covers the range $\lambda \geqslant \sup(2, 2/\rho)$.

The treatment of IE (3.1) for small λ cannot be considered directly, since it is not λ itself that must be small, but a certain parameter μ ($\mu \geqslant \lambda$) related to the geometry of Ω. Let Ω be a convex domain (the case of non-convex Ω is more difficult and requires the use of results from [2]) and let a be the minimum radius of curvature of its contour L. Define μ by $\mu = \lambda / a$. We have $\mu = \lambda$ only when Ω is a circle.

Let Ω_0 and Ω_ε be domains, defined, respectively as the loci of the points in Ω whose distances from the contour along the normal are at least $a(1-0)$ and $a(1-\varepsilon)$, $\varepsilon > 0$. For small values of μ one can construct a degenerate solution of IE (3.1) in the form

$$\varphi(x,y) = \frac{1}{2\pi\lambda^3}\left\{\iint_\Omega f(\xi,\eta) l\left(\frac{R}{\lambda}\right) d\xi d\eta + O\left[\exp\left(-\frac{1-\varepsilon}{\mu}\gamma_2\right)\right]\right\} \tag{3.7}$$

$$(x,y)\in\Omega_\varepsilon, \quad l(t) = \int_0^\infty \frac{u^2}{L(u)} J_0(ut)\,du$$

The last integral must be interpreted in the generalized sense.

To construct a solution of the boundary-layer type in the domain $\Omega - \Omega_0$, we rewrite Eq. (3.1) in the form

$$\iint_{\Omega-\Omega_0} \varphi(\xi,\eta) k\left(\frac{R}{\lambda}\right) d\xi d\eta + \iint_{\Omega_0} \varphi(\xi,\eta) k\left(\frac{R}{\lambda}\right) d\xi d\eta = 2\pi\lambda f(x,y) \tag{3.8}$$

and consider points $(x,\ y) \in \Omega - \Omega_\varepsilon$. We get the following estimate for the last integral in (3.8)

$$\iint_{\Omega_0} \varphi(\xi,\eta) k\left(\frac{R}{\lambda}\right) d\xi d\eta = O\left[\frac{1}{\mu}\exp\left(-\frac{\varepsilon}{\mu}\gamma_1\right)\right] \tag{3.9}$$

Draw the normal from a point $A(x,\ y) \in \Omega - \Omega_0$ to L. Let the length of the normal be n and its point of intersection with the contour $B(x_1,\ y_1)$. Take a point $O(x_0,\ y_0)$ on L as reference point and measure the distance s between the points O and B along L The numbers n and s will be the new coordinates of A in the curvilinear system of coordinates $(n,\ s)$ Provided that $-l/2 < s \leqslant l/2$ (where l is the perimeter of L), each pair of numbers (x, y) in the domain $\Omega - \Omega_0$ will correspond to just one pair of numbers (n, s) and conversely.

By (3.8), the IE (3.8), written in (n, s) coordinates, is

$$\int_0^{1/\mu} d\beta \int_{-k/\mu}^{k/\mu} \varphi(\beta,\gamma) k(r)\, d\gamma + O\left[\frac{1}{\mu^3}\exp\left(-\frac{\varepsilon}{\mu}\gamma_1\right)\right] = \frac{2\pi}{\lambda} f(b,c) \tag{3.10}$$

$$0 \leqslant b \leqslant 1/\mu, \quad |c| \leqslant k/\mu$$

$$b = n/\lambda, \quad c = s/\lambda, \quad r = \sqrt{(\beta-b)^2 + (\gamma-c)^2}, \quad k = l/(2a)$$

$$\varphi(\beta,\gamma) \equiv \varphi(\xi,\eta), \quad f(b,c) \equiv f(x,y)$$

Now, letting μ tend to zero in (3.10), we obtain the following IE for a boundary-layer type solution

$$\int_0^\infty d\beta \int_{-\infty}^{\infty} \varphi(\beta,\gamma) k(r)\, d\gamma = \frac{2\pi}{\lambda} f(b,c) \quad (0 \leqslant b < \infty, \ |c| < \infty) \tag{3.11}$$

Equation (3.11) may be solved in closed form by using Fourier transforms of functions of c and then applying the Wiener–Hopf method.

Define the relative thickness of the boundary layer as $H = b_0/\lambda$, where b_0 may be determined, e.g. by the condition

$$\max_c \{|\varphi(b_0, c) - \varphi^*(b_0, c)||\varphi^*(b_0, c)|^{-1}\} = 0.025 \tag{3.12}$$

with $\varphi^*(b, c)$ the principal part of $\varphi(b, c)$ as $b \to \infty$ — this is identical with $\varphi(x, y)$ when the latter is given by (3.7). The boundary layer should obviously be included in the domain $\Omega - \Omega_0$, so that the range of the parameter μ is bounded by $\mu < H^{-1}$.

In conclusion, we mention that asymptotic methods have also been used with success in dealing with various non-linear problems of mechanics with mixed boundary conditions; see, for example [13].

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